Problem Set 3 : QNMs and Schwarzschild BH

I. LINEAR PERTURBATIONS OF BLACK HOLES

We will start by studying perturbations of a static BH with spherical symmetry : the Schwarzschild BH, whose metric is

$$ds^2 = g^{(0)}_{\mu\nu}\, dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$g^{(0)}_{\mu\nu}$ being the background metric. We also make the assumption that we are in an empty space st Einstein eqs are

$$R^{(0)}_{\mu\nu} = 0.$$

Problem 1.1

We want to perturb this metric via small perturbations $h_{\mu\nu}$ st

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \qquad ; \qquad |h_{\mu\nu}| \ll 1.$$

and the associated Einstein eqs are given by

$$R_{\mu\nu} = 0.$$

Let us write this equation in terms of Christoffel symbols To this end, we introduce the inverse métrique

$$h^{\mu\nu} = g^{(0)\mu\alpha} g^{(0)\beta\nu} h_{\alpha\beta}$$

and

$$g^{\mu\nu} = g^{(0)\mu\nu} - h^{\mu\nu} + O(h^2).$$

Thus, perturbed Christoffel symbols are

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu} \right)$$

$$= \frac{1}{2} g^{(0)\rho\alpha} \left(\partial_\mu g^{(0)}_{\alpha\nu} + \partial_\nu g^{(0)}_{\alpha\mu} - \partial_\alpha g^{(0)}_{\nu\mu} \right)$$

$$+ \frac{1}{2} g^{(0)\rho\alpha} \left(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu} \right)$$

$$\underbrace{- \frac{1}{2} h^{\rho\alpha} \left(\partial_\mu g^{(0)}_{\alpha\nu} + \partial_\nu g^{(0)}_{\alpha\mu} - \partial_\alpha g^{(0)}_{\mu\nu} \right)}_{\substack{-\frac{1}{2} g^{(0)\alpha\beta} g^{(0)\rho\lambda} h_{\beta\lambda} (\dots) \\ = - g^{(0)\rho\lambda} h_{\beta\lambda} \Gamma^{(0)\beta}_{\mu\nu}}}$$

$$\Rightarrow \Gamma^\rho_{\mu\nu} = \Gamma^{(0)\rho}_{\mu\nu} + \delta \tilde{\Gamma}^\rho_{\mu\nu}$$

with $\delta \tilde{\Gamma}^\rho_{\mu\nu} = \frac{1}{2} g^{(0)\rho\alpha} (\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu}) - g^{(0)\rho\alpha} h_{\beta\alpha} \Gamma^{(0)\beta}_{\mu\nu}.$

$$= \frac{1}{2} g^{(0)\rho\alpha} \left(\nabla_\mu h_{\alpha\nu} + \nabla_\nu h_{\alpha\mu} - \nabla_\alpha h_{\mu\nu} \right)$$

that transforms as a tensor.

Indeed,

$$\nabla_\mu h_{\alpha\nu} = \partial_\mu h_{\alpha\nu} - \Gamma^{(0)\lambda}_{\mu\alpha} h_{\lambda\nu} - \Gamma^{(0)\lambda}_{\mu\nu} h_{\alpha\lambda}$$

$$\nabla_\nu h_{\alpha\mu} = \partial_\nu h_{\alpha\mu} - \Gamma^{(0)\lambda}_{\nu\alpha} h_{\lambda\mu} - \Gamma^{(0)\lambda}_{\nu\mu} h_{\alpha\lambda}$$

$$\nabla_\alpha h_{\mu\nu} = \partial_\alpha h_{\mu\nu} - \Gamma^{(0)\lambda}_{\alpha\mu} h_{\lambda\nu} - \Gamma^{(0)\lambda}_{\alpha\nu} h_{\mu\lambda}$$

Plugging it back, one has

$$g^{(0)\rho\alpha}\Big(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu} - \cancel{\Gamma^{(0)\lambda}_{\mu\alpha} h_{\lambda\nu}} - \Gamma^{(0)\lambda}_{\mu\nu} h_{\alpha\lambda}$$

$$- \cancel{\Gamma^{(0)\lambda}_{\nu\alpha} h_{\lambda\mu}} - \Gamma^{(0)\lambda}_{\nu\mu} h_{\alpha\lambda} + \cancel{\Gamma^{(0)\lambda}_{\alpha\mu} h_{\lambda\nu}} + \cancel{\Gamma^{(0)\lambda}_{\alpha\nu} h_{\mu\lambda}}\Big)$$

$$= g^{(0)\rho\alpha}\left(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu} - 2\,\Gamma^{(0)\lambda}_{\mu\nu} h_{\alpha\lambda}\right) \quad ok$$

Note that $\nabla_\mu \equiv$ at the one wrt the background metric.

Let us now derive the perturbed Ricci tensor. By definition, we know that

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}$$

and we want to write it as

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + \delta\tilde{R}_{\mu\nu}$$

We have

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^{(0)\alpha}_{\beta\alpha}\Gamma^{(0)\beta}_{\mu\nu} + \Gamma^{(0)\alpha}_{\beta\alpha}\,\delta\tilde{\Gamma}^\beta_{\mu\nu} + \delta\tilde{\Gamma}^\alpha_{\beta\alpha}\,\Gamma^{(0)\beta}_{\mu\nu}$$

$$- \Gamma^{(0)\alpha}_{\beta\nu}\Gamma^{(0)\beta}_{\mu\alpha} - \Gamma^{(0)\beta}_{\mu\alpha}\,\delta\tilde{\Gamma}^\alpha_{\beta\nu} - \Gamma^{(0)\alpha}_{\beta\nu}\,\delta\tilde{\Gamma}^\beta_{\mu\alpha}$$

$$= R^{(0)}_{\mu\nu} + \partial_\alpha\,\delta\tilde{\Gamma}^\alpha_{\mu\nu} - \partial_\nu\,\delta\tilde{\Gamma}^\alpha_{\mu\alpha} + \Gamma^{(0)\alpha}_{\beta\alpha}\,\delta\tilde{\Gamma}^\beta_{\mu\nu} + \delta\tilde{\Gamma}^\alpha_{\beta\alpha}\,\Gamma^{(0)\beta}_{\mu\nu}$$

$$- \Gamma^{(0)\beta}_{\mu\alpha}\,\delta\tilde{\Gamma}^\alpha_{\beta\nu} - \Gamma^{(0)\alpha}_{\beta\nu}\,\delta\tilde{\Gamma}^\beta_{\mu\alpha} \qquad \nabla_\alpha\,\delta\tilde{\Gamma}^\alpha_{\mu\nu} \qquad -\nabla_\nu\,\delta\tilde{\Gamma}^\alpha_{\mu\alpha}$$

$$\Rightarrow \delta\tilde{R}_{\mu\nu} = \nabla_\alpha\,\delta\tilde{\Gamma}^\alpha_{\mu\nu} - \nabla_\nu\,\delta\tilde{\Gamma}^\alpha_{\mu\alpha}.$$

$$= 0$$

$$\Rightarrow \boxed{\nabla_\nu\,\delta\tilde{\Gamma}^\alpha_{\mu\alpha} = \nabla_\alpha\,\delta\tilde{\Gamma}^\alpha_{\mu\nu}} \quad \text{Einstein eq}$$

Problem 1.2

We have scalar harmonics and we need to create an object with 1 index with spherical symmetry.

$\rightarrow$ We can only apply covariant derivatives (on the sphere) $\rightarrow D^A$; $\rightarrow D^A \varepsilon_{AB}$

For 2 indices, it's the same, we only have $\rightarrow D^A D^B$; $\rightarrow \varepsilon^{AC} D^B D_C$

More explicitly, from

$$D^2 Y_{\ell m}^{a_1 \dots a_n} = -\lambda Y^{a_1 \dots a_n} \qquad (*)$$

we can obtain

$$\begin{cases} Y_{\ell m}^{A}(\theta,\phi) = D^A Y_{\ell m}(\theta,\phi) \\ X_{\ell m}^{A}(\theta,\phi) = -\varepsilon^{AB} D_B Y_{\ell m}(\theta,\phi) \end{cases}$$

→ show that they satisfy $(*)$ and find their associated λ

Or for rank 2 tensors, we have

$$\begin{cases} \Omega^{AB} Y_{\ell m}(\theta,\phi) \\ Y_{\ell m}^{AB}(\theta,\phi) = \left(D^A D^B + \frac{1}{2}\underbrace{\ell(\ell+1)}\Omega^{AB}\right) Y_{\ell m}(\theta,\phi) \\ X_{\ell m}^{AB}(\theta,\phi) = 2\varepsilon^{(A|C} D^{B)} D_C Y_{\ell m}(\theta,\phi) \end{cases}$$

$D^A D^B Y_{\ell m}$ to which we apply D^a and for it to be a spherical harmonic, $D^2(\cdot) = -\lambda(\cdot)$ ↳ λ

We will then use them to extend the perturbation of the metric in an adapted basis. Let $x^a = (t,r)$, we have

$$\begin{cases} h_{ab} = \sum_{\ell,m} a_{ab}^{\ell m}(x^a)\, Y_{\ell m}(x^A) & \text{(scalaire)} \\ h_{aA} = \sum_{\ell,m} b_a^{\ell m}(x^a)\, Y_A^{\ell m}(x^A) + c_a^{\ell m}(x^a)\, X_A^{\ell m}(x^A) & \text{(vector)} \\ h_{AB} = \sum_{\ell,m} d^{\ell m}(x^a)\, \Omega_{AB} Y_{\ell m}(x^A) + e_{\ell m}(x^a)\, Y_{AB}^{\ell m}(x^A) + f_{\ell m}(x^a)\, X_{AB}^{\ell m}(x^A) & \text{(tenseur)} \end{cases}$$

We can divide this basis into two parts: axial ($\equiv$ odd modes) and polar ($\equiv$ even modes)

II. REGGE-WHEELER EQUATION

Axial perturbations $\equiv$ odd $\rightarrow f_{\ell m} = 0$ (R-W gauge)

Problem 2.1

We have

$$\begin{cases} h_{ab}^{ax.} = 0 \\ h_{aA}^{ax.} = \sum_{\ell,m} c_a^{\ell m}(x^a)\, X_A^{\ell m}(x^A) \\ h_{AB}^{ax.} = 0 \end{cases}$$

We thus set

$$c_a^{\ell m} \rightarrow \underset{\underset{h_{tA}}{\parallel}}{h_0(t,r)} \text{ and } \underset{\underset{h_{rA}}{\parallel}}{h_1(t,r)}$$

Problem 2.2

We know that

$$\Omega^{BC} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\sin^2\theta \end{pmatrix}$$

$$\varepsilon_{AC} = \begin{pmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{pmatrix}$$

$$\Rightarrow \varepsilon_A{}^B = \varepsilon_{AC}\,\Omega^{CB} = \begin{pmatrix} 0 & -1/\sin\theta \\ -\sin\theta & 0 \end{pmatrix}$$

on the sphere. Because $Y_{\ell m}$ are scalar functions, $D_B = \partial_B$ on the sphere

$$\Rightarrow \begin{cases} X_\theta^{\ell m} = -\dfrac{1}{\sin\theta}\,\partial_\phi Y_{\ell m}(\theta,\phi) \\ X_\phi^{\ell m} = \sin\theta\,\partial_\theta Y_{\ell m}(\theta,\phi) \end{cases}$$

such that non-vanishing components are

$$\begin{cases} h_{tA} = \sum_{\ell,m} h_0(t,r)\left(-\dfrac{1}{\sin\theta}\,\partial_\phi Y_{\ell m}(\theta,\phi)\,,\ \sin\theta\,\partial_\theta Y_{\ell m}(\theta,\phi)\right) \\ h_{rA} = \sum_{\ell,m} h_1(t,r)\left(-\dfrac{1}{\sin\theta}\,\partial_\phi Y_{\ell m}(\theta,\phi)\,,\ \sin\theta\,\partial_\theta Y_{\ell m}(\theta,\phi)\right) \end{cases}$$

We also remember that we can write

→ associated Legendre polynomial

$$Y_\ell^m(\theta,\phi) = N_\ell^m\,P_\ell^m(\cos\theta)\,e^{im\phi}$$

↳ azimutal part

normalisation constant

with $\begin{cases} \phi \text{ the azimutal angle} & (x,y)\text{-plane} \\ \theta \text{ the polar angle} & z\text{-plane} \end{cases}$

Moreover, we have

$$\begin{cases} \ell \geq 0 & \rightarrow \ell \nearrow\nearrow \equiv \text{complex structure} \\ -\ell \leq m \leq \ell \end{cases}$$

Thus, we obtain

$$h_{\mu\nu} = \sum_{\ell,m}\begin{pmatrix} 0 & 0 & -im\,h_0\,\dfrac{Y_{\ell m}}{\sin\theta} & h_0\,\sin\theta\,\partial_\theta Y_{\ell m} \\ 0 & 0 & -im\,h_1\,\dfrac{Y_{\ell m}}{\sin\theta} & h_1\,\sin\theta\,\partial_\theta Y_{\ell m} \\ -im\,h_0(t,r)\,\dfrac{Y_{\ell m}(\theta,\phi)}{\sin\theta} & -im\,h_1\,\dfrac{Y_{\ell m}}{\sin\theta} & 0 & 0 \\ h_0(t,r)\,\sin\theta\,\partial_\theta Y_{\ell m} & h_1\,\sin\theta\,\partial_\theta Y_{\ell m} & 0 & 0 \end{pmatrix}$$

Problem 2.3

Any m dependency disappeared → results from spherical symmetry (of hydrogen atom)

III. ZERILLI EQUATION

See mathematica notebook